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On equidistribution properties of Hecke eigenforms

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1 Introduction.

This is a brief survey of some of the recent progress on equidistribution properties of Hecke eigenforms on arithmetic surfaces. For simplicity let $\Gamma = SL(2, \mathbf{Z})$ be the full modular group and $X_\Gamma = \Gamma \backslash \mathbf{H}$ be the corresponding modular surface, where \mathbf{H} denotes the upper half plane. On X_Γ we have the (normalized) invariant measure

$$d\mu = \frac{1}{\text{area}(X_\Gamma)} \frac{dx dy}{y^2},$$

associated to the Poincaré metric $y^{-2}(dx^2 + dy^2)$. For integer $k \geq 8$, denote by $S_{2k}(\Gamma)$ the space of holomorphic cusp forms of weight $2k$ with respect to Γ . It is well-known that $S_{2k}(\Gamma)$ is a finite-dimensional Hilbert space with respect to the Petersson scalar product. Define $J_k = \dim_{\mathbf{C}} S_{2k}(\Gamma)$, and recall that by the Riemann-Roch theorem we have

$$J_k \sim k/6$$

as $k \rightarrow \infty$. Let $\{f_{k,j}\}_{1 \leq j \leq J_k}$ be the orthonormal basis of Hecke eigenforms in $S_{2k}(\Gamma)$. Each $f_{k,j}$ gives rise to a new probability measure on $S_{2k}(\Gamma)$:

$$d\mu_{k,j} = y^{2k} |f_{k,j}|^2 d\mu.$$

As an analogue of the quantum unique ergodicity conjecture for the Maass-Hecke eigenforms [Sar], one can formulate the following equidistribution conjecture for the holomorphic Hecke eigenforms:

Conjecture. For any compact region $A \subset X_\Gamma$, we have

$$\lim_{k \rightarrow \infty} \int_A d\mu_{k,j} = \int_A d\mu \quad (1)$$

This is equivalent to

$$\lim_{k \rightarrow \infty} \int_{X_\Gamma} \phi d\mu_{k,j} = \int_{X_\Gamma} \phi d\mu \quad (2)$$

for any Schwarz function $\phi \in \mathcal{S}(X_\Gamma)$.

Let's first examine the depth and the arithmetic implication of this conjecture.

Take ϕ to be an even Maass-Hecke eigenform of eigenvalue $\lambda_\phi = \frac{1}{4} + t_\phi^2$, then

$$\int_{X_\Gamma} \phi d\mu = 0 .$$

On the other hand, Harris-Kudla [HK] and Watson [Wa] proved, by means of the theta correspondence and Siegel-Weil formula, that

$$\left| \int_{X_\Gamma} \phi d\mu_{k,j} \right|^2 = \frac{\Lambda(1/2, \text{sym}^2(f_{k,j}) \otimes \phi) \Lambda(1/2, \phi)}{\Lambda(1, \text{sym}^2(f_{k,j}))^2 \Lambda(1, \text{sym}^2(\phi))}, \quad (3)$$

where

$$\begin{aligned} \Lambda(s, \phi) &= \pi^{-s} \Gamma\left(\frac{s+it_\phi}{2}\right) \Gamma\left(\frac{s-it_\phi}{2}\right) L(s, \phi), \\ \Lambda(s, \text{sym}^2(\phi)) &= \pi^{-3s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+2it_\phi}{2}\right) \Gamma\left(\frac{s-2it_\phi}{2}\right) L(s, \text{sym}^2(\phi)), \\ \Lambda(s, f_{k,j}) &= (2\pi)^{-s} \Gamma(s + (2k-1)/2) L(s, f), \\ \Lambda(s, \text{sym}^2(f_{k,j})) &= \pi^{-3s/2} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+2k-1}{2}\right) \Gamma\left(\frac{s+2k}{2}\right) L(s, \text{sym}^2(f_{k,j})), \\ \Lambda(s, \text{sym}^2(f_{k,j}) \otimes \phi) &= \pi^{-3s} \Gamma\left(\frac{s+2k-1+it_\phi}{2}\right) \Gamma\left(\frac{s+2k-1-it_\phi}{2}\right) \\ &\times \Gamma\left(\frac{s+2k+it_\phi}{2}\right) \Gamma\left(\frac{s+2k-it_\phi}{2}\right) \Gamma\left(\frac{s+1+it_\phi}{2}\right) \Gamma\left(\frac{s+1-it_\phi}{2}\right) L(s, \text{sym}^2(f_{k,j}) \otimes \phi). \end{aligned}$$

Thus, (2) boils down to the subconvexity bound

$$L(1/2, \text{sym}^2(f_{k,j}) \otimes \phi) = o(k)$$

as $k \rightarrow \infty$, while currently the best bound we know is only

$$L(1/2, \text{sym}^2(f_{k,j}) \otimes \phi) = O_{\phi, \epsilon}(k^{1+\epsilon}),$$

which follows from the Phragmén-Lindelöf convexity principle. The Generalized Riemann Hypothesis would implies

$$L(1/2, \text{sym}^2(f_{k,j}) \otimes \phi) = O_{\phi, \epsilon}(k^\epsilon),$$

which in turn predicts the rate of convergence for the equidistribution

$$\int_{X_\Gamma} \phi d\mu_{k,j} = O_{\phi, \epsilon}(k^{-1/2+\epsilon}). \quad (4)$$

2 A theorem of Shiffman-Zelditch

Shiffman-Zelditch [SZ] proved the following theorem:

Theorem (Shiffman-Zelditch): There exists a full density subsequence of $\{f_{k,j}\}_{1 \leq j \leq J_k, k \geq 8}$ such that (1) holds, i.e. there exist a subset $\Lambda_k \subset \{1, \dots, J_k\}$ satisfying

$$\lim_{k \rightarrow \infty} \frac{\#\Lambda_k}{J_k} = 1,$$

such that for any compact region $A \subset X_\Gamma$, we have

$$\lim_{k \rightarrow \infty, j \in \Lambda_k} \int_A d\mu_{k,j} = \int_A d\mu. \quad (5)$$

Moreover using the potential theory, they showed that the zeros of the sequence $f_{k,j}$, $j \in \Lambda_k$ are also equidistributed:

$$\lim_{k \rightarrow \infty, j \in \Lambda_k} \frac{\#\{z \in A, f_{k,j}(z) = 0\}}{J_k} = \int_A d\mu. \quad (6)$$

Note by the Riemann-Roch theorem, we have

$$\#\{z \in X_\Gamma, f_{k,j}(z) = 0\} \sim J_k, \quad \text{as } k \rightarrow \infty.$$

If we define $L = T^*(X_\Gamma)$, the cotangent bundle to X_Γ , and denote by X_Γ^* the compactification of X_Γ , then we have the interpretation of $S_{2k}(\Gamma)$ as the space of holomorphic sections of $2k$ -th tensor power of L with vanishing condition at the cusp:

$$S_{2k}(\Gamma) = H_{cusp}^0(X_\Gamma^*, L^{2k}).$$

In this context, the theorem of Shiffman-Zelditch has an analogue for holomorphic sections of tensor powers of any ample Hermitian line bundle L on a compact Kähler manifold X of higher dimension. For details see [SZ].

3 Bergman kernel

Define a new probability measure on X_Γ by

$$d\mu_k = \frac{\sum_{j=1}^{J_k} y^{2k} |f_{k,j}(z)|^2}{J_k} d\mu,$$

which is an average of the measures $d\mu_{k,j}$, $1 \leq j \leq J_k$. As a first step towards the conjecture (1), we proved the following theorem [L]:

Theorem. For any measurable subset A on the modular surface X , we have

$$\lim_{k \rightarrow \infty} \int_A d\mu_k = \int_A d\mu. \quad (7)$$

In fact, for any $\epsilon > 0$,

$$\int_A d\mu_k = \int_A d\mu + O_\epsilon(k^{-1/2+\epsilon}) \quad (8)$$

holds uniformly for all A on X .

The Hecke operator $T_k(m)$ ($m \geq 1$) acts on cusp form $f \in S_{2k}(\Gamma)$ by

$$T_k(m)f = \frac{1}{n} \sum_{ad=m} a^{2k} \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right).$$

$T_k(m)$ can be represented by the holomorphic automorphic kernel $C_k^{-1}m^{2k-1}h_{k,m}(z, z')$ (C_k defined in (12) below),

$$h_{k,m}(z, z') = \sum_{ad-bc=m} (cz' + dz' + az + b)^{-2k}, \quad (9)$$

where the sum is taken over all integer matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant m , in the sense that

$$\langle f, C_k^{-1}m^{2k-1}h_{k,m}(\cdot, -\bar{z}') \rangle_k = (T_k(m)f)(z'), \quad (10)$$

where \langle, \rangle_k is the (normalized) Petersson inner product on $S_{2k}(\Gamma)$. This kernel was first studied by Petersson, and later used by Zagier to give a new proof the Eichler-Selberg trace formula for the Hecke operators.

The series in (9) is absolutely convergent and $h_{k,m}(z, z')$ as a function of each variable z or z' is a cusp form in $S_{2k}(\Gamma)$, and we have the identity

$$C_k^{-1}m^{2k-1}h_{k,m}(z, z') = \sum_{j=1}^{J_k} \lambda_{j,k}(m) f_{j,k}(z) f_{j,k}(z'), \quad (11)$$

where $\lambda_{j,k}(m)$ is the Hecke eigenvalue of $f_{j,k}$ under $T_k(m)$ and

$$C_k = \frac{3(-1)^k}{2^{(2k-3)}(2k-1)}. \quad (12)$$

In particular, for $m = 1$ and $z' = z$, we obtain

$$C_k^{-1}h_{k,1}(z, -\bar{z}) = \sum_{j=1}^{J_k} |f_{j,k}(z)|^2. \quad (13)$$

Let χ_A denote the characteristic function of A on X . One can extend it (with the same notation) to \mathbf{H} as a Γ -invariant function. We have

$$\begin{aligned} \int_A d\mu_k &= \frac{1}{J_k} \int_X \chi_A(z) \left(\sum_{j=1}^{J_k} y^{2k} |f_{j,k}(z)|^2 \right) d\mu \\ &= \frac{1}{J_k C_k} \int_X \chi_A(z) h_1(z, -\bar{z}) y^{2k} d\mu \\ &= \frac{1}{J_k C_k} \int_X \chi_A(z) \left(\sum_{ad-bc=1} \frac{y^{2k}}{(c|z|^2 + d\bar{z} - az - b)^{2k}} \right) d\mu. \end{aligned} \quad (14)$$

Since replacing z by γz ($\gamma \in \Gamma$) in each term of the sum in (14) amounts to replacing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{by} \quad \gamma^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma,$$

we may decompose the sum into Γ -invariant pieces with $a + d = t$, $t \in \mathbf{Z}$. Thus,

$$\int_A d\mu_k = \sum_{t=-\infty}^{\infty} \frac{1}{J_k C_k} \int_X \chi_A(z) \left(\sum_{ad-bc=1, a+d=t} \frac{y^{2k}}{(c|z|^2 + d\bar{z} - az - b)^{2k}} \right) d\mu. \quad (15)$$

There is a bijection between the integral matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant 1 and trace t , and the set of integral binary quadratic forms g with discriminant $\text{disc}(g) = t^2 - 4$. The bijection is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g(u, v) = cu^2 + (d-a)uv - bv^2. \quad (16)$$

$$g(u, v) = \alpha u^2 + \beta uv + \gamma v^2 \mapsto \begin{pmatrix} (t-\beta)/2 & -\gamma \\ \alpha & (t+\beta)/2 \end{pmatrix}. \quad (17)$$

For $g(u, v) = \alpha u^2 + \beta uv + \gamma v^2$ and $z = x + iy$, set

$$R_g(z, t) = \frac{y^{2k}}{(\alpha(x^2 + y^2) + \beta x + \gamma - ity)^{2k}}, \quad (18)$$

then for $\gamma \in \Gamma$ we have

$$R_{\gamma^T g \gamma}(z, t) = R_g(\gamma z, t), \quad (19)$$

and (15) can be written as

$$\int_A d\mu_k = \sum_{t=-\infty}^{\infty} \frac{1}{J_k C_k} \int_X \chi_A(z) \left(\sum_{\text{disc}(g)=t^2-4} R_g(z, t) \right) d\mu, \quad (20)$$

where the sum is taken over all forms of discriminant $t^2 - 4$.

For each discriminant $D = t^2 - 4$ and a quadratic form g of discriminant D , we let γ_g to denote the isotropy group of elements leaving g fixed, and observe that

$$\sum_{\text{disc}(g)=D} R_g(z, t) = \sum_{\text{disc}(g)=D, \text{ mod } \Gamma} \sum_{\gamma \in \Gamma_g \backslash \Gamma} R_{\gamma^T g \gamma}(z, t) = \sum_{\text{disc}(g)=D, \text{ mod } \Gamma} \sum_{\gamma \in \Gamma_g \backslash \Gamma} R_g(\gamma z, t), \quad (21)$$

where $\text{mod } \Gamma$ means the sum is taken over a set of representatives for classes of quadratic forms with discriminant D . For $D \neq 0$, recall the class number $h(D)$ is finite, and thus we obtain

$$\int_X \chi_A(z) \left(\sum_{\text{disc}(g)=D} R_g(z, t) \right) d\mu = \sum_{\text{disc}(g)=D, \text{mod } \Gamma} \int_{X_g} \chi_A(z) R_g(z, t) d\mu, \quad (22)$$

where

$$X_g = \bigcup_{\gamma \in \Gamma_g \backslash \Gamma} \gamma X, \quad (23)$$

is a fundamental domain for the action of Γ_g on \mathbf{H} with X identified with a fundamental domain of Γ .

Denote by I_{elliptic} , $I_{\text{hyperbolic}}$ and $I_{\text{parabolic}}$ the corresponding contributions from those terms with $D = t^2 - 4 < 0$, $D = t^2 - 4 > 0$, and $D = t^2 - 4 = 0$ respectively. Then we compute (see [L])

$$I_{\text{elliptic}} = O(k^{-1/2+\epsilon}), \quad I_{\text{hyperbolic}} = O((4/5)^k), \quad I_{\text{parabolic}} = \int_A d\mu + O(k^{-1/2+\epsilon}),$$

and conclude that

$$\int_A d\mu_k = I_{\text{elliptic}} + I_{\text{hyperbolic}} + I_{\text{parabolic}} = \int_A d\mu + O_\epsilon(k^{-1/2+\epsilon}).$$

4 Asymptotics for the co-variance

For $\phi \in \mathcal{S}(X_\Gamma)$, define

$$\mu_{k,j}(\phi) = \int_{X_\Gamma} \phi d\mu_{k,j}, \quad \mu(\phi) = \int_{X_\Gamma} \phi d\mu.$$

In [LS], we computed asymptotically the variance for the equidistribution. For $\phi \in \mathcal{S}(X_\Gamma)$, we showed

$$\sum_{k \leq K} \sum_{j=1}^{J_k} |\mu_{f_{k,j}}(\phi) - \mu(\phi)|^2 \sim C_\phi K,$$

as $K \rightarrow \infty$. Actually we studied a smoothed version of the above sum.

Theorem (Luo-Sarnak). Fix $u \in C_0^\infty(0, \infty)$. There is a non-negative Hermitian form B defined on $\mathcal{S}(X_\Gamma)$ such that for $\phi, \psi \in \mathcal{S}(X_\Gamma)$ and any $\epsilon > 0$,

$$\begin{aligned} & \sum_{k \geq 1} u\left(\frac{2k-1}{K}\right) \sum_{j=1}^{J_k} L(1, \text{sym}^2(f_{k,j})) (\mu_{f_{k,j}}(\phi) - \mu(\phi)) \overline{(\mu_{f_{k,j}}(\psi) - \mu(\psi))} \\ &= B(\phi, \psi) \left(\int_0^\infty u(t) dt \right) K + O_{\epsilon, \psi}(K^{1/2+\epsilon}), \end{aligned}$$

as $K \rightarrow \infty$. The Laplacian Δ and the Hecke operators T_n are self-adjoint with respect to B , i.e.

$$B(\Delta\phi, \psi) = B(\phi, \Delta\psi),$$

and

$$B(T_n\phi, \psi) = B(\psi, T_n\psi).$$

Moreover for two Maass-Hecke cusp forms ϕ and ψ , normalized so that their first Fourier coefficients are 1, we have the relation

$$B(\phi, \psi) = \langle \phi, \psi \rangle L(1/2, \phi),$$

where $\langle \cdot, \cdot \rangle$ is the Petersson scalar product.

Since (see [I2])

$$k^{-\epsilon} \ll_\epsilon L(1, \text{sym}^2(f_{k,j})) \ll_\epsilon k^\epsilon,$$

for any $\epsilon > 0$, our theorem indicates that on the average, $(\mu_{f_{k,j}}(\phi) - \mu(\phi))$ has the size of $k^{-1/2}$. Also as a by-product, we obtain a new proof of the fact that $L(1/2, \phi) \geq 0$ for Maass-Hecke cusp forms ϕ .

Our proof uses Poincaré series and trace formula, involving subtle analysis of the sum of the Salié sums and the Neumann series for J -Bessel functions of large orders. For details see [LS].

5 Linnik problem, an analogy

There is a striking analogy between the equidistributions problem for Hecke eigenforms and the Linnik problem for integer points on the spheres and the Heegner points on X_Γ . For n square-free and satisfying $-n \not\equiv 1 \pmod{8}$, consider

$$V_n = \{m/|m| \in S^2, m \in \mathbf{Z}^3, |m|^2 = n\},$$

where $|m|$ is the usual Euclidean norm. A result of Gauss provides an exact formula for the number of integer points lying on the sphere $x_1^2 + x_2^2 + x_3^2 = n$,

$$\#V_n = \frac{24h(d_n)}{w_n} \left[1 - \left(\frac{d_n}{2} \right) \right],$$

where d_n , $h(d_n)$ and w_n are the discriminant, the class number and the number of units of $\mathbf{Q}(\sqrt{-n})$ respectively. Recall by Siegel's theorem,

$$n^{1/2-\epsilon} \ll_\epsilon h(-n) \ll_\epsilon n^{1/2+\epsilon}.$$

To establish that the points in V_n are equidistributed as $n \rightarrow \infty$, by Weyl's criterion, we need to show that the Weyl sum

$$W_u(n) = \frac{1}{\#V_n} \sum_{x \in V_n} u(x) = o(1), \quad (24)$$

for any spherical harmonics $u(x)$ of degree $l \geq 1$. Now $a(n) = n^{l/2} \#V_n W_u(n)$ is the n -th Fourier coefficient of the theta series

$$\theta(z, u) = \sum_{m \in \mathbf{Z}^3} u(m) e(z|m|^2),$$

which is a holomorphic cusp form for $\Gamma_0(4)$ of half-integral weight $l + 3/2$. Without loss of generality we may assume $\theta(z, u)$ is a Hecke eigenform. By Waldspurger's formula, we have

$$|a(n)|^2 = cn^{l+\frac{1}{2}} L(1/2, f \otimes \chi_{d_n}), \quad (25)$$

where f is the Shimura lift of $\theta(z, u)$ to $S_{2l+2}(\Gamma)$, χ_{d_n} is the quadratic character associated to the field $\mathbf{Q}(\sqrt{-n})$, and c is a constant depending only on $\theta(z, u)$ and f . Thus the resolution of Linnik's problem would result from any improvement of the convexity bound for the central L -values of the quadratic twists of f ,

$$L(1/2, f \otimes \chi_{d_n}) = O_\epsilon(n^{1/2+\epsilon}).$$

Linnik's problem was solved by Duke [D] in 1988 based on Iwaniec's subconvexity bound [I1] for $L(1/2, f \otimes \chi_{d_n})$,

$$L(1/2, f \otimes \chi_{d_n}) = O_\epsilon(n^{3/7+\epsilon}).$$

Similarly Duke [D] showed that the Heegner points on X_Γ are also equidistributed.

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